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LETTER TO THE EDITOR

Explicit l-reduced hierarchy of the KP hierarchy

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Abstract. In this letter we propose an interesting and meaningful conjecture: for every integer $l \geqslant 2$ there exists an l-reduced hierarchy in the form of $(D_1 D_{m+l} + P_l(D_1, D_2, \ldots, D_{l-1}, D_{l+1})D_m)\tau \cdot \tau = 0$ $(l \nmid m)$ of the bilinear KP hierarchy. The first five P_l are presented and proved by the use of the Wronskian technique.

Studies by the Kyoto group on soliton theory reveal that the KP hierarchy plays a fundamental role in the classification of soliton equations [1, 2]. A variety of soliton equations with physical interests can be reduced from this hierarchy [3, 4]. Sato discovered that the KP hierarchy in the bilinear forms in terms of Schur polynomials is nothing but the Plücker relations appearing in the theory of Grassmann manifolds provided that the τ function is expressed by the Wronskian [1, 5]. Recently, Newell [6] and Hu and Li[7] found that several famous hierarchies of soliton equations, such as the KdV, AKNS and classical Boussinesq hierarchies which can be generated by the 2-reduction of the KP hierarchy and its multicomponent analogue, have very simple bilinear forms. So it is natural to ask if for every integer $l \geqslant 2$ there exists an l-reduced hierarchy which has a simple and unified bilinear form. This problem can be expressed in a more clear way as a conjecture:

There exists the following type of l-reduced hierarchy

$$(D_1 D_{m+l} + P_l(D_1, D_2, \ldots, D_{l-1}, D_{l+1})D_m)\tau \cdot \tau = 0 \qquad m \in \mathbb{Z}_+, l \nmid m \tag{1}$$

or furthermore,

$$\left(D_1 D_{m+l} - \frac{1}{l+1} D_{l+1} D_m + Q_l(D_1, D_2, \ldots, D_{l-1}) D_m\right)\tau \cdot \tau = 0 \qquad m \in \mathbb{Z}_+, l \nmid m \tag{1'}$$

of the bilinear KP hierarchy for every integer $l \geqslant 2$, where P_l, Q_l are polynomials and the bilinear operator $D_k^i D_n^j$ is defined as follows [8]

$$D_k^i D_n^j a(t) \cdot b(t) = (\partial_{t_k} - \partial_{t'_k})^i (\partial_{t_n} - \partial_{t'_n})^j a(t) b(t')|_{t'=t}$$

$$t = (t_1, t_2, \ldots) \qquad t' = (t'_1, t'_2, \ldots).$$

we find that Q_l exists for $2 \leqslant l \leqslant 6$:

$$Q_2 = -\tfrac{1}{6} D_1^3$$

$$Q_3 = -\tfrac{1}{4} D_1^2 D_2$$

$$Q_4 = -\tfrac{1}{120}(D_1^5 + 15 D_1 D_2^2 + 20 D_1^2 D_3)$$

$$Q_5 = -\tfrac{1}{48}(D_1^4 D_2 + D_2^3 + 8D_1D_2D_3 + 6D_1^2D_4)$$

$$Q_6 = -\tfrac{1}{5040}(D_1^7 + 105D_1^3D_2^2 + 70D_1^4D_3 + 210D_2^2D_3 + 280D_1D_3^2 + 630D_1D_2D_4 + 504D_1^2D_5).$$

These facts can be obtained by the Wronskian technique [9, 10]. It has been proved for every $2 \leqslant l \leqslant 6$ that if f_i $(1 \leqslant i \leqslant N)$ satisfies

$$\partial_{t_m} f_i = \partial_{t_1}^m f_i \qquad m \in \mathbb{Z}_+,\ l \nmid m \tag{2a}$$

$$\partial_{t_1}^l f_i = \lambda_i f_i \tag{2b}$$

then $W(f_1, f_2, \ldots, f_N)$ satisfies (1). The procedure for proving this is quite similar, though the computation is more complicated as l increases. So here we only use the case $l = 3$ as an illustration.

For simplicity, in the following we shall use the abbreviated notation of Freeman and Nimmo [9] for the Wronskian and its derivatives and a new modified version

$$((\hat{N})_k, r_1, \ldots, r_l) \equiv \begin{cases} (\widehat{k-1}, k+1, \ldots, N, r_1, \ldots, r_l) & 0 \leqslant k \leqslant N \\ 0 \qquad k < 0 \text{ or } k > N. \end{cases}$$

The proof can be completed by the following four steps:

(i) Use (2a) and the following determinantal identity [9]

$$\partial_x |a_1(x), \ldots, a_n(x)| = \sum_{i=1}^{n} |a_1(x), \ldots, \partial_x a_i(x), \ldots, a_n(x)|$$

where $a_i(x)$ $(1 \leqslant i \leqslant n)$ is an n-column vector, to calculate the derivatives of τ. For example,

$$\tau(\widehat{N-1}) \qquad \tau_{t_1} = (\widehat{N-2}, N) \qquad \tau_{t_2} = -(\widehat{N-3}, N-1, N) + (\widehat{N-2}, N+1)$$

$$\tau_{t_1t_1} = (\widehat{N-3}, N-1, N) + (\widehat{N-2}, N+1) \qquad \tau_{t_m} = -\sum_k (-1)^{N+k}((\widehat{N-1})_k, k+m)$$

$$\tau_{t_1t_m} = -\sum_k (-1)^{N+k}((\widehat{N-2})_k, N, k+m) + (\widehat{N-2}, N+m).$$

(ii) Rewrite (1) as a one-degree bilinear form expressed by D_{m+1} and D_m. For $l = 3$, we know that

$$(D_1D_{m+3} - \tfrac{1}{4}D_4D_m - \tfrac{1}{4}D_1^2D_2D_m)\tau \cdot \tau$$
$$= 2D_{m+3}\tau_{t_1} \cdot \tau + D_m[-\tfrac{1}{2}(\tau_{t_4} + \tau_{t_1t_1t_2}) \cdot \tau + \tau_{t_1t_2} \cdot \tau_{t_1} + \tfrac{1}{2}\tau_{t_1t_1} \cdot \tau_{t_2}].$$

(iii) Substitute the expressions in (i) of the derivatives of τ into the expression in (ii) of (1), then transform the terms in (1) by some Jacobi-type determinantal identities which can be derived by the use of the Laplace expansion theorem. For $l = 3$, we need the following:

$$|Dab||Dcd| - |Dac||Dbd| + |Dad||Dbc| = 0 \qquad [9]$$

$$|Haef||Hbcd| - |Hbef||Hacd| + |Hcef||Habd| - |Hdef||Habc| = 0$$

where D is an $n\times(n-2)$ matrix, H is an $n\times(n-3)$ matrix, a, b, c, d, e and f are n-column vectors. Then we know that

$$\begin{aligned}(D_1D_{m+3}-\tfrac{1}{4}D_4D_m-\tfrac{1}{4}D_1^2D_2D_m)\tau\cdot\tau\\ =\sum_k(-1)^{N+k}\{&[2(\widehat{N-2},k+m+3)+(\widehat{N-5},N-3,N-2,N-1,k+m)\\ &-(\widehat{N-4},N-2,N,k+m)+(\widehat{N-3},N+1,k+m)](\hat{N})_k\\ &+(\widehat{N-2},k+m)[-((\widehat{N-1})_k,N+3)+((\widehat{N-2})_k,N,N+2)\\ &-((\widehat{N-3})_k,N-1,N,N+1)]\}\\ &-(\widehat{N-2},N+m+3)(\widehat{N-1})-(\widehat{N-2},N+m)(\widehat{N-2},N+2)\\ &+(\widehat{N-2},N+m+2)(\widehat{N-2},N)-(\widehat{N-2},N+m-1)\\ &\times(\widehat{N-3},N,N+1)+(\widehat{N-2},N+m)(\widehat{N-3},N-1,N+1)\\ &-(\widehat{N-2},N+m+1)(\widehat{N-3},N-1,N).\end{aligned}$$

(iv) Use (2b) and the following determinantal identity [9]

$$\left(\sum_{i=1}^{n}\lambda_i\right)|a_1,\ldots,a_n|=\sum_{i=1}^{n}|a_1,\ldots,\lambda a_i,\ldots,a_n|$$

where a_i $(1\leqslant i\leqslant n)$ is an n-column vector and λa_i denotes $(\lambda_1a_{i1},\ldots,\lambda_na_{in})^T$, to prove the expression in (iii) of (1) equals zero. For $l=3$, we know that

$$\begin{aligned}\left(\sum_{i=1}^{N}\lambda_i\right)(\widehat{N-2},k+m)\\ =(\widehat{N-5},N-3,N-2,N-1,k+m)-(\widehat{N-4},N-2,N,k+m)\\ +(\widehat{N-3},N+1,k+m)+(\widehat{N-2},k+m+3)\qquad(k\geqslant0)\end{aligned}$$

$$\left(\sum_{i=1}^{N}\lambda_i\right)(\hat{N})_k=\begin{cases}(\hat{N})_{k-3}+((\widehat{N-3})_k,N-1,N,N+1)-((\widehat{N-2})_k,N,N+2)\\ \quad+((\widehat{N-1})_k,N+3)\qquad(0\leqslant k\leqslant N-2)\\ (\hat{N})_{k-3}+((\widehat{N-1})_k,N+3)-(\widehat{N-3},N,N+1)\qquad(k=N-1)\\ (\hat{N})_{k-3}-(\widehat{N-3},N-1,N+1)+(\widehat{N-2},N+2)\qquad(k=N).\end{cases}$$

Similar natures also exist in the MKP, BKP hierarchies and their multicomponent analogues.

We hope that this conjecture can be proved and the general expression of the P_l (or Q_l) can be found in the future.

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